A VEHICLE ROUTING PROBLEM WITH STOCHASTIC DEMAND

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We consider a natural probabilistic variation of the classical vehicle routing problem (VRP), in which demands are stochastic. Given only a probabilistic description of the demand we need to design routes for the VRP. Motivated by applications in strategic planning and distribution systems, rather than resolving the problem when the demand becomes known, we propose to construct an a priori sequence among all customers of minimal expected total length. We analyze the problem using a variety of theoretical approaches. We find closed-form expressions and algorithms to compute the expected length of an a priori sequence under general probabilistic assumptions. Based on these expressions we find upper and lower bounds for the probabilistic VRP and the VRP re-optimization strategy, in which we find the optimal route at every instance. We propose heuristics and analyze their worst case performance as well as their average behavior using techniques from probabilistic analysis. Our results suggest that our approach is a strong and useful alternative to the strategy of re-optimization in capacitated routing problems.

The deterministic vehicle routing problem (VRP) is a well studied problem in the operations research literature. In this paper, we study an important variation of the VRP, in which demands are probabilistic in nature rather than deterministic. In particular, a single vehicle of limited capacity must meet demands at n fixed locations, returning periodically to the depot to empty its current load. The objective is to minimize the total distance traveled. In this paper, we consider the situation in which demand at each location is unknown at the time when the tour is designed, but is assumed to follow a known probability distribution. This situation arises in practice whenever a company (e.g., UPS), on any given day, is faced with the problem of deliveries (collections) to (from) a set of customers, which have random demand.

An obvious approach to this problem is to redesign the routes when the demand becomes known. There are, however, several difficulties with this approach; the system's operator might not have the resources for doing so; or, it may be that such redesign of tours is not sufficiently important to justify the required effort and cost. Even more importantly, the operator may have other priorities, such as regularity and personalization of service, by having the same vehicle and driver visit a particular customer every day. Moreover, it might be very difficult to learn the demand on a particular day before actually visiting the customer.

Instead of redesigning the routes every day we propose a different strategy to update the routes: Determine a fixed a priori sequence among all potential customers. Depending on when information about a customer's demand becomes available we can define two different strategies for updating the routes.

Strategy a

Under Strategy **a** the vehicle visits all the customers in the same fixed order as under the a priori sequence, but serves only customers requiring service that day. The total expected distance traveled corresponds to the fixed length of the a priori sequence plus the expected value of the additional distance that must be covered whenever the demand on the sequence exceeds vehicle capacity.

Strategy b

Strategy **b** is defined similarly to Strategy **a** with the sole difference that customers with no demand on a particular instance of the vehicle route are simply skipped. To illustrate the difference between the two strategies consider the example in Figure 1. If the a priori sequence is (0, 1, 2, 3, 4, 5, 6, 0), the depot is node 0, the vehicle has capacity 3, and the demand of the customers is $D_1 = 0$, $D_2 = 2$, $D_3 = 1$, $D_4 = 0$, $D_5 = 2$, $D_6 = 0$, then under Strategy **a** the resulting routes are shown in Figure 1 (left) and under Strategy **b** the resulting routes are shown in Figure 1 (right). Note that at node 3 the capacity is reached and the vehicle is forced to return to the depot.

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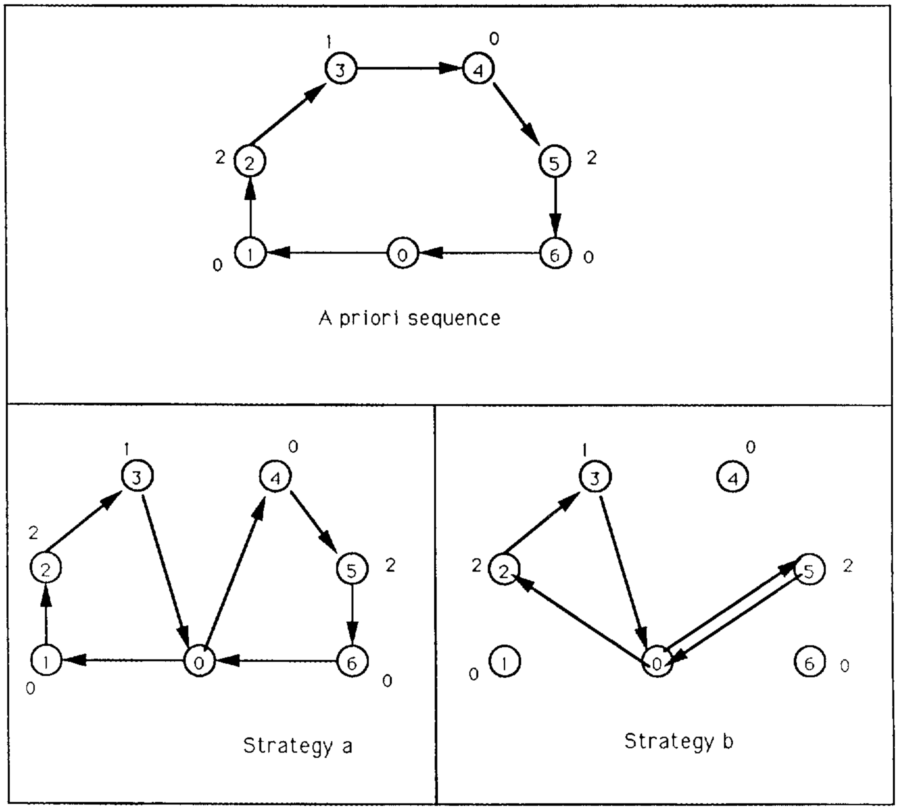

Figure 1. The two updating strategies **a** and **b**.

There is an important difference in the philosophy of the two updating strategies. Strategy **a** models situations in which the demand (if any) of any particular customer becomes known only when the customer is visited. The vehicle is then forced to return to the depot when its capacity is reached. Under Strategy **b**, however, the actual demand is known before the tour starts (customers call or the operator calls them or in the case of package deliveries the addresses are known), so that savings can occur by skipping customer locations with zero demand. Of course, in Strategy **b** we assume that the demand is known before the vehicle starts its route, so it is clearly better to use a re-optimization strategy. As mentioned, however, there might be practical considerations that make the re-optimization strategy less attractive (computing resources are not available, it is time consuming even if resources are available, regularity of service, etc.).

The natural question that arises is how to choose the a priori sequence. We propose to choose an a priori sequence of minimal expected total length, which corresponds to the expected total length of the fixed set of routes plus the expected value of the extra distance that might be required by a particular realization of the demand. The extra distances will be due to the fact that demand on the route may occasionally exceed the capacity of the vehicle and force it to go back to the depot before continuing on its route. We call the problem of selecting the a priori sequence of minimum expected length, when we use updating Strategy **a** (**b**), the probabilistic vehicle routing problem (PVRP) under Strategy **a** (**b**). We now review some situations in which vehicle routing problems with stochastic demand arise.

Application Areas

In a strategic planning scenario, consider a delivery and collection company which has decided to begin service in a particular area. The company has carried out a market survey and identified a number n of potential major customers who during any collection/distribution period have a significant probability of requiring a visit. The company wishes to estimate the resources necessary to serve these customers. At this stage of planning, the company assigns the same probability distribution of demand to all potential customers. To address the planning problem the company will wish to estimate approximately the expected amount of travel that will be necessary on a typical day to serve the subset of n customers that will require a visit.

In a routing context, consider, for example, a problem in which a central bank has to collect money on a daily basis from several but not all of its branches. The capacity Q of the vehicle used may not correspond to any physical constraint but to an upper bound on the amount of money that a vehicle might carry for safety reasons. The distribution of demand at each particular branch may be different depending on the amount of money it handles. In the same way, there is a similar problem when the bank wishes to deliver money to automatic teller machines that are located in several locations in each area.

Similarly, the distribution of packages from a post office can be modeled as a PVRP, where the probability that a certain building requires a visit is given and the capacity Q corresponds to the physical constraint that a truck can carry only a fixed weight or volume. Other examples reported in the literature include a "hot meals" delivery system (Bartholdi et al. 1983) and routing of forklifts in a cargo terminal or in a warehouse.

Brief Literature Review

The scientific literature concerning the VRP has been expanding at a very rapid pace, see, for example, the three excellent review volumes on the traveling salesman problem (Lawler et al. 1985), on routing and scheduling (Bodin et al. 1983) and on vehicle routing (Golden and Assad 1988), each of which offers several hundreds of references. Except for an isolated result in the 1970s (Tillman 1969), VRPs with stochastic elements in their definitions have received attention only recently. Stewart and Golden (1983), Dror and Trudeau (1986), Laporte and Louveau (1987), and Laporte, Louveau and Mercure (1987) use techniques from stochastic programming to solve optimally small problems and find bounds for the problems. Compared with this approach, our approach is entirely different. We propose to find an a priori solution for the problem, which is easily updated when the demand is realized. Moreover, we derive worst case and average case bounds for the performance of the strategy we propose. We also compare the strategy of finding the a priori solution with the re-optimization strategy, which is the best one can possibly do.

The idea of using an a priori sequence for the solution of traveling salesman problems when instances are modified probabilistically was first introduced in the Ph.D. thesis of Jaillet (1985) (see also Jaillet 1988). This idea was generalized to other combinatorial optimization problems in the Ph.D. thesis of Bertsimas (1988), in which the probabilistic minimum spanning tree, the probabilistic traveling salesman problem, the probabilistic vehicle routing problem, and facility location problems were analyzed. In all these investigations the demand distribution is assumed to be binary, i.e., customer i has a unit demand with probability p_i or does not have any demand with probability $1 - p_i$. In this paper, we consider arbitrary discrete-demand distributions.

The paper is organized as follows. In Section 1, we introduce the problem formally and establish the notation. In Section 2, we address the question of finding closed-form expressions and algorithms to compute the expected length of an a priori sequence under general probabilistic assumptions. In Section 3, we prove upper and lower bounds for the PVRP and the VRP re-optimization strategy, in which we find the optimal route after the demand is realized. We further use these bounds to propose heuristics with provable worst case performance. In Section 4, we examine the asymptotic behavior of the PVRP using techniques from probabilistic analysis for the case that customer locations are randomly distributed in the Euclidean plane. In the final section, we summarize the contributions of the paper and discuss the limitations and applicability of our model as well as future research.

1. FORMAL DEFINITION AND NOTATION

In this section, we formally define the PVRP and establish the notation we will use. Given a complete network, let the nodes be $\{0, 1, \ldots, n\}$, where node 0 denotes the depot and the set $V = \{1, 2, \ldots, n\}$ denotes the set of customer locations. The distances $d(i, j)$ are assumed to be symmetric (although our results can easily be modified to hold even in the nonsymmetric case) and they satisfy the triangle inequality: $d(i, j) \leq d(i, k) + d(k, j)$.

Let the capacity of the vehicle be Q (Q is an integer) and let D_i, $i = 1, \ldots, n$ be the random variable that describes the demand of customer i. We assume that the probability distribution of D_i is discrete and known. Let $p_i(k) = \Pr\{D_i = k\}$, $i = 1, \ldots, n$ and $k = 0, 1, \ldots, K$. We further assume that $K \leq Q$, that is, no single location has demand exceeding the capacity Q. We further assume that the demands are independent.

There are $(K + 1)^n$ possible realizations of the demand and therefore $(K+1)^n$ possible instances of the problem. If we solve the underlying VRP optimally at every problem instance, i.e., we find the route that minimizes the total distance traveled, let $R_{VRP}(D_1, D_2, \ldots, D_n)$ be the optimal route length if the demand is $D_1, D_2, \ldots, D_n$. Note that since the demand is stochastic this is a random variable. We call the expectation of this random variable the expected length under the re-optimization strategy, because we redesign (re-optimize) the routes at every problem instance. This expected length is thus given by

$$E[R_{VRP}] = \sum_{i_1, \ldots, i_n} p_1(i_1) \ldots p_n(i_n) R_{VRP}(i_1, \ldots, i_n), \qquad (1)$$

where the summation is over all demand instances for the nodes. Clearly, the exact estimation of $E[R_{VRP}]$ is a very difficult problem because it involves $(K+1)^n$ terms. Moreover, in order to evaluate each term $(R_{VRP}(i_1, \ldots, i_n))$ we need to solve exactly a VRP. So, in a strategic planning scenario, in which a company needs to have an estimate of the expected travel cost, the expected length of the re-optimization strategy is not a realistic alternative computationally.

Related to the vehicle routing re-optimization strategy, we can define a traveling salesman re-optimization strategy in which, at every instance of the problem, the vehicle visits all the customers with nonzero demand according to the optimal traveling salesman tour. We denote

$$E[R_{TSP}] = \sum_{S \subseteq I} \prod_{i \in S} \Pr\{D_i > 0\} \prod_{i \notin S} \Pr\{D_i = 0\} L_{TSP}(S), \qquad (2)$$

where $L_{TSP}(S)$ is the length of the optimal TSP tour among customers in the set S.

Let us now consider the two a priori strategies we are proposing. Given an a priori sequence τ let $L_\tau^i(i_1, \ldots, i_n)$ be the length of the a priori sequence τ that will result under strategy $i = a, b$ if the demand pattern is $i_1, \ldots, i_n$. We denote with

$$E[L_\tau^a] = \sum_{i_1, \ldots, i_n} p_1(i_1) \ldots p_n(i_n) L_\tau^a(i_1, \ldots, i_n), \qquad (3)$$

the expected length of the a priori sequence τ under Strategy **a** and

$$E[L_\tau^b] = \sum_{i_1, \ldots, i_n} p_1(i_1) \ldots p_n(i_n) L_\tau^b(i_1, \ldots, i_n), \qquad (4)$$

the expected length of the a priori sequence τ under Strategy **b**.

Our goal is to find the a priori sequences τ_a and τ_b that minimize the expected lengths in (3) and (4), respectively. Although there are $(K+1)^n$ terms in both (3) and (4) we will be able to compute the expected length of an a priori sequence efficiently in the next section.

2. THE EXPECTED LENGTH OF AN A PRIORI SEQUENCE

In this section, we propose an algorithm for finding the expected length of an a priori sequence under Strategies **a** and **b**. We first consider the case of binary demand, where a customer either has a demand of one unit or it does not have any demand.

2.1. Binary Demand

We first examine the important special case in which all the demand is binary, either 0 or 1. There are several motivations for examining this case separately. From an applications viewpoint, it is important in situations in which the randomness in the demand can be modeled by the presence (or not) of a customer. For example, in the distribution of packages by the post office, there is a potential set of customers each requiring a visit with probability p_i.

From a theoretical viewpoint, the usual traveling salesman problem can be viewed as a special case of the PVRP under Strategy **a**, if the capacity $Q = n$, i.e., the problem is uncapacitated. Moreover, the probabilistic traveling salesman problem (PTSP) introduced and analyzed in Jaillet (1988) and further explored in Bertsimas (1988) can be viewed as a special case of the PVRP under Strategy **b**, for which the capacity Q is equal to n, i.e., the capacity of the vehicle is not a binding constraint. Moreover, the insights gained from the binary case carry over to the general case.

Our initial goal then is to compute $E[L_\tau^a]$, $E[L_\tau^b]$ efficiently for a given a priori sequence τ. Let p_i be the probability that customer i has a demand of one unit and $1 - p_i$ of not having any demand independently of any other customer. Then we can compute the expected length of an a priori sequence as follows.

Theorem 1. *If the a priori sequence is* $\tau = (0, 1, \ldots, n, n+1 \triangleq 0)$, *then*

$$E[L_\tau^a] = \sum_{i=0}^{n} d(i, i+1) + \sum_{i=1}^{n} \gamma_i s(i, i+1), \qquad (5)$$

$$E[L_\tau^b] = \sum_{i=1}^{n} d(0, i)p_i \prod_{r=1}^{i-1} (1 - p_r)$$
$$+ \sum_{i=1}^{n} d(i, 0)p_i \prod_{r=i+1}^{n} (1 - p_r)$$
$$+ \sum_{i=1}^{n} \sum_{j=i+1}^{n} d(i, j)p_i p_j \prod_{r=i+1}^{j-1} (1 - p_r)$$
$$+ \sum_{i=1}^{n} \sum_{j=i+1}^{n} s(i, j)\gamma_i p_j \prod_{r=i+1}^{j-1} (1 - p_r), \qquad (6)$$

where

$$s(i, j) \triangleq d(i, 0) + d(0, j) - d(i, j),$$
$$\gamma_i = 0, \quad i = 0, \ldots, Q - 1,$$
$$\gamma_i = p_i \sum_{k=1}^{\lfloor i/Q \rfloor} f(i - 1, kQ - 1), \; i \geq Q, \qquad (7)$$

and $f(m, r) \triangleq Pr\{$*exactly* r *customers among the customers* $1, \ldots, m$ *have nonzero demand*$\}$ *are computed from the recursion: For* $m = 1, \ldots, n$, $r = 1, \ldots, m$

$$f(m, r) = p_m f(m - 1, r - 1)$$
$$+ (1 - p_m)f(m - 1, r), \qquad (8)$$

with the initial conditions

$$f(m, m) = \prod_{i=1}^{m} p_i, \quad f(m, 0) = \prod_{i=1}^{m} (1 - p_i).$$

Proof. Consider first Strategy **a**. The expected length of the sequence is a summation of the length of the a priori sequence plus the expected value of the extra distance when the vehicle reaches its capacity. To evaluate this second term, let i be a node on the sequence, where the vehicle reaches its capacity. The vehicle will then go to the depot before going back to the following node in the route, which is $i + 1$ under Strategy **a**, even if node $i + 1$ has no demand. The extra distance traveled is then $s(i, i + 1) = d(i, 0) + d(0, i + 1) - d(i, i + 1)$.

In (5), γ_i is the probability that the vehicle reaches its capacity Q at node i. Clearly, $\gamma_i = 0$ for $i = 0, \ldots, Q - 1$. Consider now node i. For the vehicle to reach its capacity at node i ($i \geq Q$), node i must have a unit demand and from the previous $i - 1$ nodes exactly $kQ - 1$ must be present for some $k = 1, \ldots, \lfloor i/Q \rfloor$, so that the capacity is reached with the addition of node i. From this observation (7) follows. The probabilities $f(m, r)$ are computed recursively from (8) by conditioning on the event that node m has a demand.

Under Strategy **b**, the first three terms in (6) are simply the expected length of the tour τ in the probabilistic traveling salesman sense. In particular, the distance $d(i, j)$ contributes to the expectation if nodes i and j are present and all the intermediate nodes have zero demand. The fourth term is identical with Strategy **a**, except that when the vehicle reaches its capacity at node i, it goes back, after a visit to the depot, to the first node j with a nonzero demand, skipping nodes $i + 1, i + 2, \ldots, j - 1$ with no demand.

As an application of (5) and (6) we find the closed-form expressions derived in Jaillet and Odoni (1988) for the case in which all points have the same probability p of requiring a visit. Then expressions (8) imply that $f(m, r) = \binom{m}{r}p^r(1 - p)^{m-r}$, and thus

$$E[L_\tau^a] = \sum_{i=0}^{n} d(i, i + 1) + \sum_{i=1}^{n} s(i, i + 1) \cdot \sum_{k=1}^{\lfloor i/Q \rfloor} \binom{i - 1}{kQ - 1} p^{kQ}(1 - p)^{i-kQ},$$

$$E[L_\tau^b] = E[L_\tau] + \sum_{i=1}^{n} \sum_{k=1}^{\lfloor i/Q \rfloor} \binom{i - 1}{kQ - 1} p^{kQ}(1 - p)^{i-kQ} \cdot \sum_{j=i+1}^{n} s(i, j)p(1 - p)^{j-i-1},$$

where $E[L_\tau]$ denotes the expected length of the a priori tour τ in the probabilistic traveling salesman sense (the first three terms in (6)).

An important consequence of (5) and (6) is that they provide an algorithm of $O(n^2)$ to compute $E[L_\tau^a]$, $E[L_\tau^b]$ for the general case of unequal probabilities, because the computation of the probabilities $f(m, r)$ can be done recursively from (8) in $O(n^2)$, and there are $n - Q$ nonzero probabilities γ_i. The computation of each of these probabilities from (7) requires the evaluation of a sum of at most $\lceil n/Q \rceil$ terms. Thus, we can compute all the γ_i in $O((n - Q)n/Q + n^2) = O(n^2)$. Finally, the expectation of the length of the sequence, given that we have already computed the probabilities γ_i is done in $O(n)$ for Strategy **a** and $O(n^2)$ for Strategy **b**, which means that we can compute the expected length of an a priori sequence in $O(n^2)$ for both strategies. In the next subsection, we generalize these expressions for the case of general discrete-demand distributions.

2.2. General Demand

In this section, we find expressions for the expected length of a priori sequences under Strategies **a** and **b**.

Let $\Pr[D_i = k] = p_i(k)$ be the probability that the demand of customer i, D_i, is equal to k, for k ranging from 0 to K. As mentioned, we assume that $K < Q$, i.e., the vehicle's capacity is greater than the largest demand of a customer on a given day. This assumption removes the consideration of multiple returns to the depot from the same node.

Similarly, as before, we define γ_i to be the probability that the vehicle *exactly* reaches its capacity at node i and δ_i to be the probability that the vehicle exceeds its capacity at node i. Then the expected length of an a priori sequence is computed as follows.

Theorem 2. *If the a priori sequence is* $\tau = (0, 1, \ldots, n, n+1 \triangleq 0)$ *then,*

$$E[L_\tau^a] = \sum_{i=0}^{n} d(i, i+1) + \sum_{i=1}^{n} [\delta_i s(i, i) + \gamma_i s(i, i+1)], \qquad (9)$$

$$\begin{aligned} E[L_\tau^b] = {} & \sum_{i=1}^{n} d(0, i) p_i \prod_{r=1}^{i-1} (1 - p_r) \\ & + \sum_{i=1}^{n} d(i, 0) p_i \prod_{r=i+1}^{n} (1 - p_r) \\ & + \sum_{i=1}^{n} \sum_{j=i+1}^{n} d(i, j) p_i p_j \prod_{r=i+1}^{j-1} (1 - p_r) \\ & + \sum_{i=1}^{n} \left(\delta_i s(i, i) + \sum_{j=i+1}^{n} \gamma_i p_j s(i, j) \cdot \prod_{r=i+1}^{j-1} (1 - p_r) \right), \end{aligned} \qquad (10)$$

where

$$p_r = \sum_{k=1}^{K} p_r(k) = 1 - p_r(0),$$

$$s(i, j) = d(i, 0) + d(0, j) - d(i, j),$$

$$\gamma_1 = 0$$

$$\gamma_i = \sum_{q=1}^{\lfloor iK/Q \rfloor} \left\{ \sum_{k=1}^{K} p_i(k) f(i-1, qQ-k) \right\}, \qquad 2 \leq i \leq n,$$

$$\delta_1 = 0,$$

$$\delta_i = \sum_{q=1}^{\lfloor iK/Q \rfloor} \left\{ \sum_{k=1}^{K-1} \left(\sum_{r=k+1}^{K} p_i(r) \right) f(i-1, qQ-k) \right\}, \qquad 2 \leq i \leq n,$$

and $f(m, r) = \Pr\{$*the total demand of the customers* $1, \ldots, m$ *is* $r\}$ *are computed from the recursion*:

$$f(m, r) = \sum_{k=0}^{\min\{K, r\}} p_m(k) f(m-1, r-k), \qquad m = 2 \ldots n,\ r = 0 \ldots Km$$

with the initial conditions:

$$f(m, r) = 0 \quad \text{for } r > Km,$$

$$f(1, r) = \begin{cases} p_1(r) & \text{for } 0 \leq r \leq K, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Under both Strategies **a** or **b**, the expressions are almost the same as in Theorem 1 with the exception that two cases have to be distinguished—whether the capacity is reached exactly at node i or not. In particular, the explanation of the different terms appearing in these expressions still holds as in Theorem 1.

As in the case of binary demand, these expressions lead to a $O(K^2 n^2)$ algorithm to compute the expected sequences under both Strategies **a** and **b**. Finally, a closed-form expression for $f(m, r)$ can be found when all points have the same probability $p_i(k) \triangleq q_k$ of requiring a visit. Then with the interpretation that r_i is the number of customers with demand i we get

$$f(m, r) = \begin{cases} \displaystyle\sum_{\substack{\{r_1 + 2r_2 + \cdots + Kr_K = r \\ r_0 + r_1 + \cdots + r_K = m\}}} \binom{m}{r_0, r_1, \ldots, r_K} q_0^{r_0} q_1^{r_1} \cdots q_K^{r_K} & 0 \leq r \leq mK,\ m \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\binom{m}{r_0, r_1, \ldots, r_K}$ denotes the multinomial coefficient.

3. BOUNDS AND APPROXIMATIONS FOR THE PVRP

In this section, we derive upper and lower bounds on the various strategies we consider with the ultimate goal of comparing these strategies from a worst and an average case perspective, and we propose heuristics with good worst and average case performances. We will be concerned with the case of general demand distribution.

Let τ_a, τ_b be the optimal sequences for Strategies **a** and **b**, respectively, of the PVRP and let τ_p, τ_{TSP} be the optimal tours for the PTSP and the TSP, respectively. For $Q = n$ and in the case of binary demand, clearly $\tau_a = \tau_{TSP}$, $\tau_b = \tau_p$.

3.1. Relation Among the Different Strategies

We first concentrate on understanding the relation among the expected lengths of the optimal solutions for the PVRP under Strategies **a** and **b**, ($E[L^a_{\tau a}]$, $E[L^a_{\tau b}]$), the expected length of the optimal tour for the PTSP, ($E[L_{\tau_p}]$), the length of the optimal deterministic tour (L_{TSP}), and the expectation of the re-optimization strategies, $E[R_{VRP}]$, $E[R_{TSP}]$.

Proposition 1. *For the case of arbitrary demand and under the triangle inequality*

$$E[R_{VRP}] \leqslant E[L^b_{\tau_b}] \leqslant E[L^a_{\tau_a}]. \tag{11}$$

Proof. Consider an a priori sequence τ. Then $L^b_\tau(i_1, \ldots, i_n) \leqslant L^a_\tau(i_1, \ldots, i_n)$, because under Strategy **b** we skip customers with zero demand and because of the triangle inequality the length of the resulting route is smaller. Note that the breakpoints in the routes occur at the same nodes under both strategies. As a result,

$$E[L^b_\tau] \leqslant E[L^a_\tau].$$

Consider now the optimal a priori sequence τ_a under Strategy **a**. The above inequality gives $E[L^b_{\tau_a}] \leqslant E[L^a_{\tau_a}]$. But, because of the optimality of the sequence τ_b for Strategy **b**, $E[L^b_{\tau_b}] \leqslant E[L^b_{\tau_a}]$, from which the right inequality of (11) follows. Also, since $R(i_1, \ldots, i_n) \leqslant L^b_{\tau_b}(i_1, \ldots, i_n)$ in every instance the left inequality follows.

3.2. Lower Bounds

In this subsection, we derive lower bounds for the different strategies. For convenience, we assume that the distance matrix is symmetric.

Proposition 2. *Under the triangle inequality*

$$E[R_{VRP}] \geqslant \max\left[\frac{2}{Q}\sum_{r=1}^{n} d(0,r)E[D_r], E[R_{TSP}]\right], \tag{12}$$

$$E[L^a_{\tau_a}] \geqslant \max\left(\frac{2}{Q}\sum_{r=1}^{n} d(0,r)E[D_r], L_{TSP}\right), \tag{13}$$

$$E[L^b_{\tau_b}] \geqslant \max\left(\frac{2}{Q}\sum_{r=1}^{n} \mathrm{d}(0,\mathrm{r})\mathrm{E}[\mathrm{D}_r], \mathrm{E}[\mathrm{L}_{\tau_p}]\right). \tag{14}$$

Proof. Consider an instance $S = S(i_1, \ldots, i_n)$ of the problem that depends upon the demand pattern $i_1, \ldots, i_n$. Under the re-optimization strategy, a vehicle starts from the depot, visits a subset X_j of customers ($|X_j| \leqslant Q$), returns to the depot, and then continues to the next subset X_{j+1}. Then, if L_j is the length of the route for visiting the subset X_j of customers in the optimal solution at instance S, then the optimal length $R(i_1, \ldots, i_n) = \sum_{j=1}^{I(i_1, \ldots, i_n)} L_j$, where $I(i_1, \ldots, i_n)$ is the number of subtours in this particular instance. Clearly

$$L_j \geqslant 2d(0,r) \quad \text{for all } r \in X_j.$$

Multiplying by demand i_r and summing over all nodes in X_j gives

$$L_j \sum_{r\in X_j} i_r > 2 \sum_{r\in X_j} d(0,r)i_r.$$

Now since the capacity is never exceeded on any subtour, we have $\sum_{r\in X_j} i_r \leqslant Q$, and so it follows that

$$L_j \geqslant \frac{2}{Q}\sum_{r\in X_j} d(0,r)i_r.$$

Then from (1), we have

$$E[R_{VRP}] \geqslant \frac{2}{Q}\sum_{i_1,\ldots,i_n} p_1(i_1)\ldots p_n(i_n) \cdot \sum_{j=1}^{I(i_1,\ldots,i_n)} \sum_{r\in X_j} d(0,r)i_r$$

$$= \frac{2}{Q}\sum_{r=1}^{n} d(0,r)E[D_r],$$

which establishes the first inequality in (12). In addition, since at every instance the length $R_{VRP}(i_1, \ldots, i_n) \geqslant L_{TSP}(i_1, \ldots, i_n)$, i.e., the length of the vehicle routing re-optimization strategy is larger than the TSP re-optimization strategy, we have that

$$E[R_{VRP}] \geqslant E[R_{TSP}].$$

Moreover, from the triangle inequality, $s(i,j) \geqslant 0$. Therefore, from (9) $E[L^a_{\tau_a}] \geqslant L_{\tau_a} \geqslant L_{TSP}$. Similarly, $E[L^a_{\tau_b}] \geqslant E[L_{\tau_b}] \geqslant E[L_{\tau_p}]$, and hence, using (11) we obtain (13) and (14).

In Proposition 2, if the distance matrix is asymmetric, then we should replace the term

$$\frac{2}{Q}\sum_{r=1}^{n} d(0,r)E[D_r]$$

with

$$\frac{1}{Q}\sum_{r=1}^{n} [d(0,r)+d(r,0)]E[D_r].$$

3.3. Upper Bounds

In this subsection, we concentrate on finding upper bounds for the two probabilistic Strategies **a** and **b**. We first consider the case of identically distributed demand and for convenience we assume that the distance matrix is symmetric. We consider the cyclic heuristic introduced in Haimovitch and Rinnooy Kan (1985) in the context of the deterministic VRP.

Cylic Heuristic

Step 1. Given an initial sequence $\tau \triangleq \tau_1 = (0, 1, 2, \ldots, n, 0)$, consider the sequences

$\tau_i = (0, i, \ldots, n, 1, \ldots, i-1, 0), i = 2, \ldots n.$

Step 2. *Compute* $E[L^a_{\tau_i}]$ for all $i = 1, \ldots, n$.

Step 3. The sequence with the minimum expected length among $E[L^a_{\tau_i}]$, $i = 1, \ldots, n$ is the proposed solution τ_H to the PVRP under Strategy **a**.

We now analyze the worst case behavior of the cyclic heuristic under the assumption that each customer has the same demand distribution.

Proposition 3. *Let D be the random variable describing the demand of each customer. If the initial sequence to the cyclic heuristic is the optimal traveling salesman tour and* τ_H *is the tour proposed by the cyclic heuristic, then under the triangle inequality*

$$E[L^a_{\tau_a}] \le E[L^a_{\tau_H}] \le L_{TSP}\left(1 - \frac{1}{n}\right) + 2\left(1 + \frac{nE[D]}{Q}\right)\frac{\sum_{i=1}^n d(0,i)}{n}. \qquad (15)$$

Proof. If the initial sequence to the cyclic heuristic is the optimal deterministic tour, then let

$$L \triangleq \sum_{i=1}^{n-1} d(i, i+1) + d(n, 1).$$

With this definition the lengths of the sequences τ_i become:

$$L_{\tau_1} = L + d(0,1) + d(0,n) - d(1,n),$$

$$L_{\tau_i} = L + d(0,i) + d(0,i-1) - d(i,i-1), i = 2, \ldots n.$$

As a result,

$$\sum_{i=1}^n L_{\tau_i} = 2\sum_{i=1}^n d(0, i) + (n-1)L.$$

Clearly,

$$E[L^a_{\tau_a}] \le E[L^a_{\tau_H}] \le \frac{1}{n}\sum_{i=1}^n E[L^a_{\tau_i}].$$

From (9) and the triangle inequality

$$E[L^a_{\tau_i}] \le L_{\tau_i} + 2\sum_{j=1}^n (\gamma_j + \delta_j)\, d(0, j(i)),$$

where $j(i) = (j + i - 2)mod(n) + 1$. Therefore,

$$E[L^a_{\tau_H}] \le \frac{1}{n}\left[\sum_{i=1}^n L_{\tau_i} + 2\sum_{i=1}^n (\gamma_i + \delta_i)\sum_{i=1}^n d(0, i)\right],$$

since the probabilities $\gamma_i + \delta_i$ multiply every term $d(0, i)$ in $\sum_{i=1}^n E[L^a_{\tau_i}]$. Therefore,

$$E[L^a_{\tau_H}] \le \frac{1}{n}\left[2\sum_{i=1}^n d(0, i) + (n-1)L + 2\sum_{i=1}^n (\gamma_i + \delta_i)\sum_{i=1}^n d(0, i)\right]. \qquad (16)$$

Our goal is then to find an upper bound for the term $\sum_{i=1}^n (\gamma_i + \delta_i)$. We define the random variable:

$N \triangleq$ the number of breakpoints in the route, where the capacity is either exactly reached or exceeded. Let also $B_i \triangleq$ the indicator random variable taking the value 1 if a breakpoint occurs at customer i and 0 otherwise. With these definitions, $N = \sum_{i=1}^n B_i$. But since $\Pr\{B_i = 1\} = \gamma_i + \delta_i$, then

$$E[N] = \sum_{i=1}^n E[B_i] = \sum_{i=1}^n (\gamma_i + \delta_i).$$

If $W \triangleq$ the total demand, then

$$E[N] = E\left[\left\lfloor\frac{W}{Q}\right\rfloor\right] \le E\left[\frac{W}{Q}\right] = \frac{nE[D]}{Q},$$

and hence,

$$\sum_{i=1}^n (\gamma_i + \delta_i) \le \frac{nE[D]}{Q}. \qquad (17)$$

Using (17) in (16) and since $L \le L_{TSP}$, (15) follows.

In Proposition 3 we considered the case of identically distributed demand. In the next proposition we consider the case in which customers' demands are not necessarily identical. Unfortunately, the upper bounds are less tight.

Proposition 4. *Let* D_i *be the random variable that describes the demand of customer i. Then under the triangle inequality*

$$E[L^b_{\tau_b}] \le E[L^b_{\tau_p}] \le E[L_{\tau_p}] + 2\sum_{i=1}^n d(0, i)\min\{E[D_i], 1\}. \qquad (18)$$

$$E[L^a_{\tau_a}] \le E[L^a_{TSP}] \le L_{TSP} + 2\sum_{i=1}^n d(0, i)\min\{E[D_i]), 1\}. \qquad (19)$$

Proof. Consider the optimal tour for the PTSP $\tau_p = (0, 1, \ldots, n, 0)$ as a solution to the PVRP under

Strategy **b**. Then,

$$E[L^b_{\tau_b}] \leq E[L^b_{\tau_p}] = E[L_{\tau_p}] + \sum_{i=1}^{n} \delta_i s(i, i) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \gamma_i p_j \prod_{r=i+1}^{j-1} (1 - p_r) s(i, j).$$

Since $s(i, j) = d(i, 0) + d(0, j) - d(i, j) \leq 2d(0, i)$ from the triangle inequality we obtain

$$\sum_{j=i+1}^{n} p_j \prod_{r=i+1}^{j-1} (1 - p_r) = (1 - \prod_{r=i+1}^{n} (1 - p_r)) \leq 1,$$

$$E[L^b_{\tau_b}] \leq E[L_{\tau_p}] + 2 \sum_{i=1}^{n} (\gamma_i + \delta_i)\, d(0, i) \leq E[L_{\tau_p}] + 2 \sum_{i=1}^{n} d(0, i) \min\{E[D_i], 1\},$$

since

$$\gamma_i + \delta_i = \sum_{r=1}^{K} \Pr\{D_i \geq r\} \Pr\{W_{i-1} = lQ + Q - r \text{ for some } l\} \leq \sum_{r=1}^{K} \Pr\{D_i \geq r\} \leq E[D_i],$$

where $W_i = \sum_{r=1}^{i} D_r$ is the cumulative demand up to customer i. Also since $\gamma_i + \delta_i$ is a probability, $\gamma_i + \delta_i \leq 1$. We prove (19) by considering the optimum TSP tour as a feasible solution to the PVRP under Strategy **a** and use the same bounding technique as before.

We now find an upper bound for the vehicle routing re-optimization strategy.

Proposition 5. *Let D_i be the random variable that describes the demand of customer i. Then under the triangle inequality*

$$E[R_{VRP}] \leq E[R_{TSP}]\left(1 - \frac{2}{Q}\right) + \frac{4}{Q} \sum_{i=1}^{n} E[D_i]\, d(0, i). \tag{20}$$

Proof. Consider an instance of the problem, in which the demand pattern is $i_1, \ldots, i_n$. Let S be the set of customers with nonzero demand. We use a result of Altinkemer and Gavish (1985) that

$$R_{VRP}(i_1, \ldots, i_n) \leq L_{TSP}(S)\left(1 - \frac{2}{Q}\right) + \frac{4}{Q} \sum_{j=1}^{n} i_j d(0, j).$$

By taking expectations the result follows immediately.

3.4. Heuristics for the PVRP

In this subsection, we exploit the bounds derived in the previous section to propose some heuristics with good worst case performance. In Section 3.3 we introduced the cyclic heuristic. In the following theorem we prove that the heuristic is within a constant factor from the optimal sequence under Strategy **a**.

Theorem 3. *Assume that the demands of the customers are identically distributed. If the initial sequence given to the cyclic heuristic is the optimal TSP tour and the sequence found by the cyclic heuristic is τ_H, then under the triangle inequality*

$$\frac{E[L^a_{\tau_H}]}{E[L^a_{\tau_a}]} \leq 2 + O\left(\frac{1}{n}\right). \tag{21}$$

Proof. From (13) and (15) we obtain

$$\frac{E[L^a_{\tau_H}]}{E[L^a_{\tau_a}]} \leq \frac{L_{TSP}(1 - 1/n) + 2/n(1 + nE[D]/Q) \sum_{i=1}^{n} d(0, i)}{\max(2E[D]/Q \sum_{i=1}^{n} d(0, i), L_{TSP})} \leq 1 - \frac{1}{n} + 1 + \frac{Q}{nE[D]} \leq 2 + O\left(\frac{1}{n}\right).$$

Moreover, if $\Pr\{D = 0\} = 0$, i.e., all customers have some demand, then one can easily strengthen the previous bound to $E[L^a_{\tau_H}]/E[R_{VRP}] \leq 2 + O(1/n)$, which means that the cyclic heuristic is within a factor of two of the re-optimization strategy. Theorem 3 says that the cyclic heuristic produces a solution to the PVRP under Strategy **a**, which, for large enough n, is within a factor of two of the optimal sequence. If, instead of the optimal deterministic tour, we give the Christofides tour to the cyclic heuristic, then the guarantee will be $5/2 + O(1/n)$, and the running time of the combined heuristic (Christofides heuristic and then the cyclic heuristic) is $O(n^3)$, since the Christofides heuristic takes $O(n^3)$ and the cyclic heuristic needs the evaluation of the expected length of n sequences, each of which takes $O(n^2)$. Therefore, this combined heuristic runs in polynomial time and produces solutions which are within a constant factor of the optimal sequence under Strategy **a**.

We now investigate the more general case of not identically distributed demand.

Theorem 4. *If the demand of customer i is distributed according to the random variable D_i, then under the* *triangle inequality*

$$\frac{E[L^b_{\tau_p}]}{E[L^b_{\tau_b}]} \leq Q+1, \quad \frac{E[L^a_{TSP}]}{E[L^a_{\tau_a}]} \leq Q+1.$$

where τ_p is the optimal tour for the PTSP.

Proof. Combining Propositions 2 and 4 we obtain

$$\frac{E[L^b_{\tau_p}]}{E[L^b_{\tau_b}]} \leq \frac{E[L_{\tau_p}] + 2\sum_{i=1}^n d(0,i)E[D_i]}{\max(2/Q\sum_{i=1}^n d(0,i)E[D_i], E[L_{\tau_p}])} \leq Q+1.$$

The bounds in the case of nonidentical demand distributions are not particularly strong in the worst case (although constant for constant Q). The reason for this is that the upper bound from Proposition 4 is not very sharp. Thus, we expect that they will be much better in practice. Moreover, we have seen that the bound produced by the cyclic heuristic applied to the identical demand case is quite strong. It is tempting to conjecture that the sequence produced by the cyclic heuristic for the nonidentical demand case when initialized by the optimum TSP is within a constant factor (independent of Q) of the optimal solution under Strategy **a** as well. Indeed, our preliminary computational experience suggests that this is the case.

4. ASYMPTOTIC BEHAVIOR FOR THE PVRP IN THE RANDOM EUCLIDEAN MODEL

In this section, we investigate the asymptotic behavior of the PVRP and of the re-optimization strategy under the random Euclidean model. Let $X_1, X_2, \ldots$ be an infinite sequence of independent, identically distributed random points in the unit square and assume that the depot is at (0, 0). Let $E[r]$ be the expected distance from the origin and let $X^{(n)}$ denote the first n points of the sequence. Our goal is to find the asymptotic behavior of the expected length of the re-optimization strategy $E[R_{VRP}(X^{(n)})]$, and the expected length of the two a priori strategies

$$E[L^a_{\tau_a}(X^{(n)})], \; E[L^b_{\tau_b}(X^{(n)})].$$

Observe that these quantities are random variables because the locations of the customers are random.

Let D be the random variable that describes the demand of each customer. Let $p = \Pr\{D > 0\}$. For the PTSP the following results are known. Then with probability 1 (Jaillet 1988):

$$\lim_{n\to\infty} \frac{E[R_{TSP}(X^{(n)})]}{\sqrt{n}} = \beta_{TSP}\sqrt{p}, \tag{22}$$

$$\lim_{n\to\infty} \frac{E[L_{\tau_p}(X^{(n)})]}{\sqrt{n}} = \beta(p), \tag{23}$$

where β_{TSP} is the constant appearing in the celebrated Beardwood, Halton and Hammersley (1959) theorem and $\beta(p)$ is a constant that only depends on p for which

$$\beta_{TSP}\sqrt{p} \leq \beta(p) \leq \min[0.92\sqrt{p}, \beta_{TSP}].$$

The asymptotic behavior of the expected length of the re-optimization strategy and the PVRP depends critically on the dependence of the capacity Q on the number of customers n. This dependence is also critical for the asymptotic behavior of the VRP examined in Haimovitch and Rinnooy Kan. Let Q_n denote the capacity of the vehicle to indicate its dependence on n. We prove the following theorem.

Theorem 5. *The asymptotic behavior of the three updating strategies is:*

1. *If Q is a constant, then almost surely*

$$\lim_{n\to\infty} \frac{E[R_{VRP}(X^{(n)})]}{n} = \lim_{n\to\infty} \frac{E[L^a_{\tau_a}(X^{(n)})]}{n} = \lim_{n\to\infty} \frac{E[L^b_{\tau_b}(X^{(n)})]}{n} = \frac{2E[r]E[D]}{Q}. \tag{24}$$

2. *If* $\lim_{n\to\infty}(Q_n/\sqrt{n}) = 0$, *then almost surely*

$$\lim_{n\to\infty} \frac{Q_n E[R_{VRP}(X^{(n)})]}{n} = \lim_{n\to\infty} \frac{Q_n E[L^b_{\tau_b}(X^{(n)})]}{n} = \lim_{n\to\infty} \frac{Q_n E[L^b_{\tau_b}(X^{(n)})]}{n} = 2E[r]E[D]. \tag{25}$$

3. *If* $\lim_{n\to\infty} Q_n/\sqrt{n} = \infty$, *then almost surely*

$$\lim_{n\to\infty} \frac{E[R_{VRP}(X^{(n)})]}{\sqrt{n}} = \beta_{TSP}\sqrt{p},$$

$$\lim_{n\to\infty} \frac{E[L^a_{\tau_a}(X^{(n)})]}{\sqrt{n}} = \beta_{TSP},$$

$$\lim_{n\to\infty} \frac{E[L^b_{\tau_b}(X^{(n)})]}{\sqrt{n}} = \beta(p). \tag{26}$$

Proof. Let $\bar{r} \triangleq \sum_{i=1}^n d(0,i)/n$.

Case 1. Assume that Q is constant. Combining the bounds from Propositions 2 and 5 we obtain

$$\frac{2E[D]}{Q}\bar{r} \leq \frac{E[R_{VRP}(X^{(n)})]}{n} \leq \frac{E[R_{TSP}(X^{(n)})]}{n}\left(1 - \frac{1}{Q}\right) + \frac{2E[D]\bar{r}}{Q}.$$

But $E[r] < \infty$ implies that $\bar{r} \to E[r]$ almost surely by the strong law of large numbers. Taking limits and using (22) we obtain (24) for the re-optimization strategy. We obtain (24) for the PVRP strategies in a very similar manner using the bounds from Propositions 1, 2, and 3.

Case 2. Similarly,

$$2E[D]\bar{r} \leq \frac{Q_n}{n} E[R_{TRP}(X^{(n)})]$$
$$\leq \frac{Q_n E[R_{TSP}(X^{(n)})]}{n} + 2E[D]\bar{r} \tag{27}$$

Since with probability 1,

$$\lim_{n\to\infty} \frac{Q_n E[R_{TSP}(X^{(n)})]}{n} = \lim_{n\to\infty} \frac{Q_n}{\sqrt{n}} \frac{E[R_{TSP}(X^{(n)})]}{\sqrt{n}} = 0$$

(25) for the re-optimization strategy follows by taking limits. Again, we obtain (25) for the PVRP strategies in a very similar manner using the bounds from Propositions 1, 2, and 3.

Case 3. From Propositions 2 and 5

$$\frac{E[R_{TSP}(X^{(n)})]}{\sqrt{n}} \leq \frac{E[R_{TSP}(X^{(n)})]}{\sqrt{n}}$$
$$\leq \frac{2E[D]\bar{r}}{Q_n/\sqrt{n}} + \frac{E[R_{TSP}(X^{(n)})]}{\sqrt{n}}.$$

From (22) and since $\lim_{n\to\infty} (Q_n/\sqrt{n}) = \infty$, we obtain (26) for the re-optimization strategy by taking limits.

With regard to Strategy **a** from Propositions 2 and 3 we obtain:

$$\frac{L_{TSP}(X^{(n)})}{\sqrt{n}} \leq \frac{E[L^a_{\tau_a}(X^{(n)})]}{\sqrt{n}}$$
$$\leq \frac{2E[D]\bar{r}}{Q_n/\sqrt{n}} + \frac{2\bar{r}}{\sqrt{n}} + \frac{L_{TSP}(X^{(n)})}{\sqrt{n}}.$$

Since $\lim_{n\to\infty} L_{TSP}(X^{(n)})/\sqrt{n} = \beta_{TSP}$ almost surely and $\lim_{n\to\infty} (Q_n/\sqrt{n}) = \infty$, we obtain (26) for Strategy **a** by taking limits.

Finally, for Strategy **b** using Propositions 1, 4, and 6 we obtain:

$$E[L_{\tau_p}(X^{(n)})] \leq E[L^b_{\tau_b}(X^{(n)})] \leq E[L_{\tau_p}(X^{(n)})]$$
$$+ 2\sum_{i=1}^{n} d(0, i)(\gamma_i + \delta_i)$$
$$\leq E[L_{\tau_p}(X^{(n)})] + 2d(0, i_{\max}) \sum_{i=1}^{n} (\gamma_i + \delta_i)$$
$$\leq E[L_{\tau_p}(X^{(n)})] + 2\sqrt{2}\,\frac{nE[D]}{Q_n}.$$

Dividing with $\sqrt{n}$, taking the limit as $n \to \infty$ and using (23), we obtain (26) for Strategy **b**.

The case $Q_n = \Theta(\sqrt{n})$ is not covered in the previous theorem. The reason is that in this case neither the radial collection term, $(2E[D]n\bar{r}/Q_n)$, nor the local collection term, $E[R_{TSP}(X^{(n)})]$, dominate as was the case in Theorem 5, where the radial collection term dominated in Cases 1 and 2 and the local calculation term dominated in Case 3.

Another interesting observation is that in both Cases 1 and 2 the sequence produced by the cyclic heuristic with initial tour any tour of length $O(\sqrt{n})$ is asymptotically optimal for Strategy **a**. Moreover, for Strategy **a** the sequence produced by the cyclic heuristic with initial sequence the optimal TSP is asymptotically optimal in Case 3 as well.

In Case 3 the PTSP solves the PVRP under Strategy **b** optimally. An even more interesting consequence of Theorem 5 is that in Cases 1 and 2 both the probabilistic Strategies **a** and **b** are asymptotically equivalent to the re-optimization strategy. Furthermore, as we argued before, in these cases the tour produced by the cyclic heuristic with initial sequence any tour of length $O(\sqrt{n})$ is asymptotically equivalent to the re-optimization strategy. In Case 3, Strategy **b** is asymptotically within a constant factor from the strategy of re-optimization. Indeed, we conjecture that the constant factor is 1, i.e., Strategy **b** is asymptotically optimal. Moreover, in this case Strategy **a** is also close to the re-optimization strategy if p is large.

Finally, we only considered the case where customer locations are uniformly distributed in the unit square. Similar asymptotic theorems can be proved in the d-dimensional Euclidean space and, furthermore, for the case where the distribution of customer locations has a continuous part with density f. We chose dimension two in the exposition because the geometry is clearer, and the uniform distribution for customer locations, because it is more intuitive.

5. CONCLUDING REMARKS

In this paper, we propose a different approach to solve vehicle routing problems when the demand is stochastic. We give analytical evidence that this approach, which is based on finding an a priori sequence among the entire set of customers, performs quite well from a worst case perspective, especially if the distribution of the demand of the customers is the same. Although we propose worst case bounds for the case of nonidentical demand distributions, our bounds were not tight. Finding tighter bounds is an interesting open problem deserving further work.

We also attempt to give analytical evidence that the a priori strategies we propose for the PVRP are very close to the strategy of re-optimization, on average. In particular, we show that if customer locations are uniformly distributed in the unit square, both Strategies **a** and **b** perform asymptotically very closely to the strategy of re-optimization, which is a strong indication of the usefulness of these strategies. Furthermore, our analysis reveals some asymptotically optimal heuristics for both Strategies **a** and **b**. It is a fair conclusion to say that a priori and re-optimization strategies perform comparably, on average.

From a practical standpoint, we believe that the a priori strategies we are proposing provide a strong alternative to the strategy of re-optimization, and therefore they can be useful in practice, especially in the absence of intense computational power. Related to this point, an interesting question deserving further research is to computationally compare the a priori and re-optimization strategies.

As a final conclusion, we believe the paper demonstrates that in the context of capacitated vehicle routing problems a priori strategies (PVRP) are a serious and practical alternative to re-optimization strategies. Our investigation in Bertsimas (1988) reached the same conclusion for other combinatorial optimization problems.

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